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Radar Absorption Effect in Hypersonic Ballistic Ranges

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Musal and Blore[1] have recently proposed an explanation for the anomalously large decrease in radar cross section that occurs when $\frac{1}{2}$-in. diam spheres are observed by 4- or 8-mm wavelength radars at velocities near 13,000 fps and pressures of 10 mm Hg. This decrease is anomalous in that the plasma sheath around the sphere is far too thin to cause enough real power absorption in the lossy plasma layer. The new explanation hinges on the diffractive effect of sharp angular gradients in the plasma sheath surrounding the sphere. Some comparisons between this new theory and experiment are presented here for two different bodies and range conditions using General Motors Defense Research Laboratories (GM DRL) data.[2]

It can be shown,[1] using the techniques of physical optics, that the radar cross section of a metal sphere covered by a thin plasma layer is

$$\frac{\sigma}{\lambda^2} = \frac{\pi^3 a^2}{4\lambda^2}\left|\int_0^{4a/\lambda} (\Gamma_{TM} - \Gamma_{TE}) \times \left(1 - \frac{\lambda\chi}{4a}\right) e^{j(4\pi h/\lambda)}\, e^{-j\pi\chi[1+(h/a)]}\, d\chi\right|^2$$

where Γ_{TM} and Γ_{TE} are the *TM* and *TE* mode-reflection coefficients for a metal-backed plasma layer, given by

$$\Gamma_{TM} = \frac{\Gamma_{pTM} + e^z}{1 + \Gamma_{pTM}e^z} \qquad \Gamma_{TE} = \frac{\Gamma_{pTE} - e^z}{1 - \Gamma_{pTE}e^z}$$

$$\Gamma_{pTM} = \frac{[1 - \Omega_p^2/(1 - j\Omega_c)](1 - \chi\lambda/4a) - [(1 - \chi\lambda/4a)^2 - \Omega_p^2/(1 - j\Omega_c)]^{1/2}}{[1 - \Omega_p^2/(1 - j\Omega_c)](1 - \chi\lambda/4a] + [(1 - \chi\lambda/4a)^2 - \Omega_p^2/(1 - j\Omega_c)]^{1/2}}$$

$$\Gamma_{pTE} = \frac{(1 - \chi\lambda/4a) - [(1 - \chi\lambda/4a)^2 - \Omega_p^2/(1 - j\Omega_c)]^{1/2}}{(1 - \chi\lambda/4a) + [(1 - \chi\lambda/4a)^2 - \Omega_p^2/(1 - j\Omega_c)]^{1/2}}$$

and z is given by

$$z = -j(4\pi h/\lambda)[(1 - \chi\lambda/4a)^2 - \Omega_p^2/(1 - j\Omega_c)]^{1/2}$$

and λ is the radar wavelength, a is the radius of the sphere, and h is the thickness of plasma layer. The plasma properties are given by the normalized plasma frequency Ω_p and the normalized electron collision frequency Ω_c, which are

$$\Omega_p = \omega_p/\omega = (1/\omega)(q^2N/\epsilon m)^{1/2} \qquad \Omega_c = v_c/\omega$$

where ω is the radar angular frequency $(2\pi f)$, v_c is the electron collision frequency, ω_p is the plasma frequency, q is the electric charge carried by an electron and m is its mass, N is the electron number density, and ϵ is the capacitivity of free space. In the simplified case where the reflection coefficients of the layer are not functions of the angle of incidence, the preceding integral can be explicitly evaluated in closed form. The more interesting case of a plasma layer around a hypersonic sphere must be evaluated by numerical integration. In its present form, this equation can handle only angular plasma gradients, and radially nonuniform plasmas must be approximated by some equivalent "average" radially uniform plasma. The more general case is currently under development. Also, because of the physical optics assumptions used in its derivation the preceding equation is most accurate for large blunt bodies which have an angular plasma coverage of less than one radian.

The appropriate plasma parameters for each set of experimental results have been obtained either from the Cornell nonequilibrium bow shock program[3] or the GM DRL equilibrium flow codes,[4] appropriately modified to map streamline space into body-related shock space using the streamline-pressure relationships given in Gravolos[5] et al. For computation purposes, the actual flow fields were approximated as follows. The thickness of the plasma layer was taken to be the thickness of the overdense region of the plasma sheath around the body. The electron density and collision frequency within this overdense region were taken to be the values which existed in the stagnation region for the speed and ambient pressure considered. In order to obtain the dependence of the radar cross section on body speed, the way in which this overdense region thickens and spreads over the body as a function of speed was approximated by interpolation between sets of flow field calculations for various velocities. Using this approach, the radar cross section was calculated as a function of velocity for the ambient pressures used in the various ballistic range studies. The theoretical results are shown in Fig. 1, along with the experimental data for a 13-mm radius sphere at 10 mm Hg pressure. It can be seen that a large decrease in the radar cross section, of the order of 15 db, is obtained with the new theory. Previous theories gave approximately 3 db of absorption under these conditions. Figure 2 shows the same comparison for a

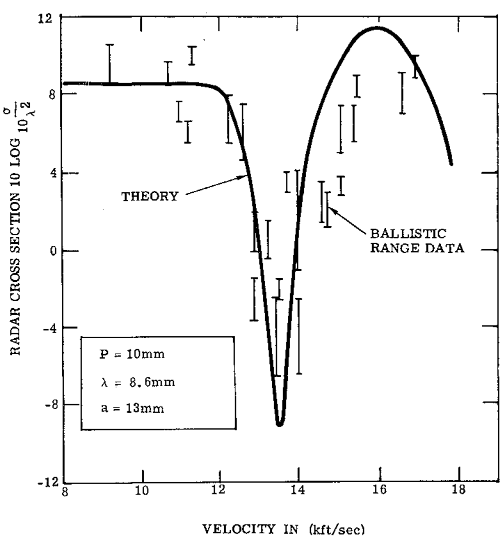

Fig. 1 Comparison between theory and experiment for the backscattering radar cross section of 13-mm radius sphere observed in a hypersonic ballistic range at 10-mm-Hg pressure vs velocity.

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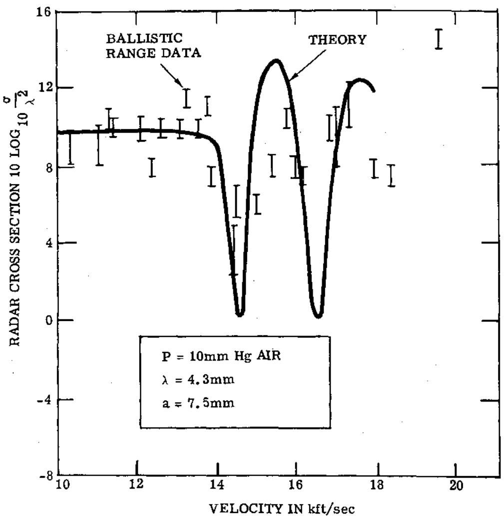

Fig. 2 Variation of radar cross section with velocity for a 7.5-mm radius sphere at 10-mm-Hg pressure; comparison of theory and experiment.

7.5-mm radius sphere at 10 mm Hg as seen by a 4.3-mm wavelength radar. The same kinds of comparisons have also been made for some previously published Canadian Armament Research and Development Establishment (CARDE) data, and similar results have been obtained. In all cases, it is evident that the theory is capable of predicting a very large decrease in cross section for these very thin plasma layers, something which was not possible with previous theories.

The details of radar cross section variation with speed depend critically on the development of the ionization around the body. In particular, we find that the results are very sensitive to the rate constants used in the nonequilibrium flow field calculations. Thus, there is a need for very good flow field calculations before exact comparisons between the theoretical and experimental results can be made. Work is continuing along the lines of improving the diffraction theory used in the prediction of the radar cross sections of plasma-covered metal bodies and in the calculation of the flow fields around blunt bodies. In conclusion, it appears that a theoretical explanation has been found for the large decrease in the radar cross section of a blunt-metal body when it is covered by a very thin plasma sheath. The decrease depends on partial coverage of the body by the plasma layer.

It is the angular gradients of the plasma properties around the body that cause severe diffraction of the radar wave. The details of the effect depend so critically upon the distribution of ionization in the plasma sheath that the effect may be useful as a flow field diagnostic.

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Lagrange Multiplier Techniques in Structural Analysis

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Nomenclature

P_i = external loads
X_j = loads between elements (internal loads)
X, P = column of these loads
$LX + P$ = 0 is the matrix expression of the equilibrium equations at nodes
$[F]$ = combined flexibility matrix, overlapped and summed
$\frac{1}{2}X'[F]X$ = strain energy of elements concerned
u = nodal deflections (column)
λ = Lagrange multipliers (column)
Δu = departure from rigid body motion (column)
u_0 = nodal deflections for unit rigid-body motions (a column for each)
$u_0\lambda$ = denotes the actual rigid body motion (λ is again a column)
δ = misfits between elements measured along the X (column)
$[K]$ = combined stiffness matrix, overlapped and summed
$\frac{1}{2}u'[K]u$ = strain energy of elements concerned
$[k]$ = a small stiffness to earth
$\frac{1}{2}\lambda' u_0'[k]u_0\lambda$ = energy stored in light springs to earth

Introduction

The Argyris methods[2] generate their matrices for solution or inversion by processes of matrix multiplication. They predominantly solve structures using forces as variables. Asplund gives the theory elegant academic form and extends it to use forces in one part of a structure and displacements in the other.[3, 4] The direct stiffness method of Turner[5] avoids matrix multiplications (except of element size) by placing the coefficients directly into the equations to be solved. The generalized stiffness solution of Jones[6] is also direct in this sense.

Dallison Method

The dual of the Jones method is the Dallison force method of 1953,[1] which should never have been forgotten. Dallison introduced element equilibrium conditions using Lagrange multipliers. If, instead, we introduce nodal equilibrium conditions using λ as Lagrange multipliers, we find that a λ_i equals the nodal displacement u_i in the direction of the applied load P_i, that is, the constant term in the equilibrium equation. This depends on work being expressible as $\Sigma u_i P_i$, as does the direct stiffness method (see Ref. 4, Sec. Ld).

A mixed Dallison method is clearly possible, in which part of the structure is analyzed using flexibilities, the other part using stiffnesses, while still retaining the characteristic of directness. The Dallison force method, using nodal equilibrium, may be expressed as

$$\begin{bmatrix} F & L' \\ L & 0 \end{bmatrix} \begin{bmatrix} X \\ u \end{bmatrix} = \begin{bmatrix} \delta \\ -P \end{bmatrix} \quad (1)$$

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